

Random walks with shrinking steps: First-passage characteristics

Tonguç Rador*

Boğaziçi University, Department of Physics, Bebek 34342, İstanbul, Turkey

Sencer Taneri†

Feza Gürsey Institute, Emek. Mah. No:68, Çengelköy 34864, İstanbul, Turkey

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We study the mean first-passage time of a one-dimensional random walker with step sizes decaying exponentially in discrete time. That is step sizes go like λ^n with $\lambda \leq 1$. We also present, for pedagogical purposes, a continuum system with a diffusion constant decaying exponentially in continuous time. Qualitatively both systems are alike in their global properties. However, the discrete case shows very rich mathematical structure, depending on the value of the shrinking parameter, such as self-repetitive and fractal-like structure for the first-passage characteristics. The results we present show that the most important quantitative behavior of the discrete case is that the support of the distribution function evolves in time in a rather complicated way in contrast to the time independent lattice structure of the ordinary random walker. We also show that there are critical values of λ defined by the equation $\lambda^K + 2\lambda^P - 2 = 0$ with $\{K, N\} \in \mathcal{N}$ where the mean first-passage time undergoes transitions.

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I. INTRODUCTION

The common model of random walk is an integral part of almost all scientific disciplines in such a way that it would not be an overstatement to call it a meta model. In this respect it is important to look for extensions and possible generalizations of the model. In fact many of these extensions are already present: the ordinary random walk is solvable in many lattices with arbitrary nearest neighbor sites, the continuum limits are known for various geometries and generalizations for space dependent diffusion constant also exist in nature.

Another likely extension of a random walk could be the case of a walker with step sizes shrinking with time. Although there is considerable work [1–6] on this in mathematical literature, the idea has not been studied extensively from a physics perspective: see [7–10]. However there is a rather straightforward motivation for it to be important. The idea is the dichotomy between a Brownian particle and an *active* random walker. In the former case the particle is merely pushed around by the surrounding molecules. That is, it is under the influence of a random external force. In the latter case however, there is a possibility for internal energy dissipation and hence the step sizes (the ability to diffuse in the medium) might decrease in time. It is a difficult exercise to visualize how this might occur in atomic scales. It is more likely that this motivation would make sense only in some form of macroscopic mean field theory.

If we can be allowed to be a little humorous, the very analogy of a *drunk man* for the ordinary random walk is in essence closer to a walker with shrinking step sizes. This example might at first seem rather nonscientific, but it brings

to mind how biological systems, with rather short memory, propagate in nature without energy intake, let it be a fish in a bowl or a fly in a room. It is most likely that step size changes in nature are not fierce under normal circumstances so their effect is immaterial. However, under extreme circumstances such as lions hunting prey, the internal energy dissipation could be important and then it is a matter of determining how the step sizes change in time from biological principles.

Nevertheless, the aim of this paper is not to make a full case for the physical relevance of a random walker with shrinking step sizes. We believe that random walk would remain a meta model with this extension as well.

The motivation for this work is the very interesting first passage characteristics of the ordinary random walk, such as the dichotomy between *certain passage* from any point and *infinite mean first passage time* from that point for a random walker in one dimension. Various processes in nature proceed by first-passage processes. For example the hunt will be over when the prey and predator meet for the first time, or a neuron will fire when the electric potential first reaches a threshold value, etc. For further information on the study of first-passage time for various systems we refer the reader to [11] (and references therein) for a review and introductory book and to [12–16] for recent literature.

The outline of the paper is as follows. In Sec. II, for purposes of motivation and developing a qualitative understanding of the problem, we consider the first-passage characteristics of a diffusion equation with a diffusion constant that decreases exponentially with time. In particular we emphasize the asymptotics of the mean first-passage time for later comparison with the discrete case.

Section III introduces the discrete random walker with shrinking step sizes with an emphasis on the most general aspects of the model. In particular we emphasize the fact that the support of the distribution evolves in time in a complicated way rendering an analytical study almost impossible.

*Electronic address: tonguc.rador@boun.edu.tr

†Electronic address: taneri@gursey.gov.tr

The first passage characteristics of the discrete random walker with exponentially decreasing step sizes are presented in Secs. IV and V. In Sec. IV the model is studied numerically for various parameters and both local and global properties of the mean first-passage time are analyzed. In Sec. V we present an in depth analysis of the global properties. In these sections we also contrast the results to the continuum model and substantiate that the two models agree on their general behaviors if not in every detail.

The continuum model and the discrete walker we study both have logarithmic divergences in the mean first-passage time as we let the position to be first passed take large values. This in turn makes it possible to have a qualitative connection of parameters: in the continuum model the diffusion constant decays exponentially in time with decay parameter τ and the correspondence takes the form $\tau \sim -1/\ln \lambda$ modulo dimensions. When we consider first-passage times of points closer to the origin we again recover the same correspondence. In short the continuum model grasps almost all the global properties of the mean first-passage times for the discrete model.

However the local behavior of the discrete walker is much richer. We have quantified **one** of its aspects as follows: we studied the mean first-passage time for general λ in comparison to the exactly solvable case for $\lambda \leq 1/2$ where the mean first-passage time from a point $\langle t(x_0) \rangle$ has the form of a ladder which increases in unit steps at the points $x_0(M) = (1 - \lambda^M)/(1 - \lambda)$ with $M \in \mathcal{N}$, that is for $\lambda \leq 1/2$ we have $\langle t(M) \rangle = M + 1$. We show that at these points, the behavior of the mean first-passage time for general λ after we subtract the ladder structure of the exactly soluble case [meaning we study $\langle t'(M) \rangle \equiv \langle t(M) \rangle - M$ as a function of λ] has transitions at λ 's defined by the equation $\lambda^K + 2\lambda^P - 2 = 0$ with $\{K, N\} \in \mathcal{N}$. As a corollary we also show that for $\lambda \leq \lambda_{12}$ the reduced mean first-passage time $\langle t'(M) \rangle$ does not depend on M and is a function of λ only. Here λ_{12} is the solution to the polynomial above for $K=1$ and $P=2$ we have $\lambda_{12} \approx 0.732$. This we believe is one of the most interesting properties of the discrete walker while there are some others we have discussed in Sec. IV.

II. A CONTINUUM EXAMPLE

For presentation purposes, before getting into the intricacies of a discrete random walk, we expose the behavior of a shrinking diffusion constant in continuum diffusion equation. That is, we consider the following:

$$\frac{\partial P}{\partial t} = D e^{-t/\tau} \frac{\partial^2 P}{\partial x^2}. \quad (1)$$

We would like to solve this in a semi-infinite line with the initial condition that $P(x, 0) = \delta(x)$ and absorbing boundary conditions at $x = x_0$, that is, $P(x_0, t) = 0$.

The differential equation in (1) can easily be solved with the above conditions to give

$$P(x, T) = \frac{1}{\sqrt{4\pi DT}} [e^{-(x^2/4DT)} - e^{-(x-2x_0)^2/4DT}], \quad (2)$$

$$T = \tau(1 - e^{-t/\tau}). \quad (3)$$

The first-passage probability to x_0 is just the flux to this point. The flux can be read from the diffusion equation if one writes it as $\partial P / \partial t = -\partial F / \partial x$. This will give

$$F(x_0, t) = \frac{x_0}{\sqrt{4\pi DT^3}} \exp\left[-\frac{x_0^2}{4DT}\right]. \quad (4)$$

From this the zeroth moment or the eventual first passing probability can be calculated to give

$$F_0(x_0) = 1 - \operatorname{erf}\left[\frac{x_0}{\sqrt{4D\tau}}\right]. \quad (5)$$

Here, $\operatorname{erf}(u) = (2/\sqrt{\pi}) \int_0^u dt e^{-t^2}$ is the error function. The result above should be contrasted to the ordinary diffusion equation where $F_0(x_0) = 1$. In the problem under consideration there is a finite first-passage probability from x_0 , however this is rapidly vanishing with increasing x_0 as one would expect.

The survival probability, that is the probability to never have been to x_0 at time t , is defined as

$$S(x_0, t) \equiv 1 - \int_0^t F(x_0, t') dt', \quad (6)$$

which gives an exact result

$$S(x_0, t) = \operatorname{erf}\left[\frac{x_0}{\sqrt{4D\tau(1 - e^{-t/\tau})}}\right]. \quad (7)$$

This predicts a finite survival probability $S(x_0, \infty) = \operatorname{erf}(x_0/\sqrt{4D\tau})$ for infinite time, again in contrast to the ordinary random walk with the survival probability vanishing as $x_0/\sqrt{4Dt}$ as $t \rightarrow \infty$.

The mean first-passage time can also be calculated. However in this case the result is only available as a series

$$\langle t(x_0) \rangle = \frac{\tau}{1 - \operatorname{erf}(u)} \sum_{n=1}^{\infty} \frac{1}{n} u^{2n} \Gamma(0.5 - n, u^2). \quad (8)$$

With $u \equiv x_0/\sqrt{4D\tau}$ and $\Gamma(a, z) \equiv \int_z^{\infty} dt e^{-t} t^{a-1}$, the incomplete γ function. To understand the behavior of this quantity we study its asymptotic behavior. The analytic properties of the error function and the incomplete γ function are readily available in textbooks. The results for the two regimes of interest are the following:

for $u \rightarrow 0$

$$\langle t(x_0) \rangle \approx \sqrt{\frac{\tau}{\pi}} \frac{x_0}{\sqrt{D}} - \frac{\pi - 2}{2\pi} \frac{x_0^2}{D} + \dots \quad (9)$$

for $u \rightarrow \infty$

$$\langle t(x_0) \rangle \rightarrow \tau \sum_{n=1}^{\infty} \frac{1}{n} - \tau^2 \frac{4D}{x_0^2} \sum_{n=1}^{\infty} 1 + \dots \quad (10)$$

So as $u \rightarrow \infty$ we expect a logarithmic divergence with a linear dependence on τ .

We are now equipped with enough intuition to attack the more difficult case of a discrete random walk with shrinking steps.

III. RANDOM WALKS WITH SHRINKING STEPS

Consider a random walker which hops equally such as to the left and right by some distance s_n at a given time n . Then the random variable for the position of the walker is simply

$$x(N) = \sum_{n=1}^N \epsilon_n s_n, \quad (11)$$

with $\epsilon_n = \pm 1$ the random variable for hopping. The mean of $x(N)$, as expected, is zero: the random walker does not propagate in the mean but rather in the standard deviation which is given by

$$\langle x^2(N) \rangle = \sum_{n=1}^N s_n^2. \quad (12)$$

From this relation it follows that if the sum converges to a constant as $N \rightarrow \infty$ there can be no continuum limit [17]. To clarify the reasoning let us remember how one takes the continuum limit. We first introduce a lattice spacing δx , which would mean that the variance becomes

$$\langle x^2(N) \rangle = \delta x^2 \sum_{n=1}^N s_n^2.$$

Now if there is a continuum limit we should be able to take $\delta x \rightarrow 0$ with $N \rightarrow \infty$ and still get a nonzero standard deviation. For example in the ordinary random walk we have $s_n = 1$, which in turn means that $\langle x^2(N) \rangle = \delta x^2 N$. So letting $N \rightarrow \infty$ can be done by $N \equiv t / \delta t$ and $\delta t \rightarrow 0$. The limit would make sense if $\delta x^2 / \delta t$ remains a constant upon taking the limits. Actually this is how one defines the diffusion constant. On the other hand if the sum in (12) converges to a constant independent of N as $N \rightarrow \infty$, we get zero standard deviation upon taking $\delta x \rightarrow 0$. Such a behavior will be present, for example, if $s_n = \lambda^n$ with $\lambda < 1$: the model which is under consideration in this paper.

This pathology is present because the support of the probability distribution function for the random variable $x(N)$ evolves in time, in sharp contrast to the ordinary random walk for which the walker always resides on a lattice site: a time independent regular structure allowing a limit for zero lattice spacing.

Disregarding this difficulty one might be tempted to assume that the use of known machinery such as Fourier transforms will be useful. However this does not generally help in practice unless special cases occur, such as infinite time limit or a particular shrinking parameter. The fact that the support of the distribution evolves in time in a generally nonuniform way will ultimately infest these approaches, especially if one is interested in the temporal evolution in the finite time regime.

For example in [7] the case $s_n = \lambda^n$ has been shown, using Fourier transforms, to be exactly solvable for $\lambda = 1/2$. This

case is exactly solvable since at each time step the support of the distribution is

$$N = 1 \quad \{-1, 1\},$$

$$N = 2 \quad \{-3/2, -1/2, 1/2, 3/2\},$$

$$N = 3 \quad \{-7/4, -5/4, -3/4, -1/4, 1/4, 3/4, 5/4, 7/4\},$$

$$\vdots \quad \vdots.$$

The support evolves, but at each time step it is a regular, equally-spaced lattice albeit with different lattice spacings for different times. For slightly more general cases, namely the infinite time limit for $\lambda = 1/2^m$, studied in [7], what happens is that the supports are actually unions of m evenly distributed support sets which results in mathematical terms to products (convolutions) of the distributions in momentum (position) space. This is the reason why these cases are easily manageable.

IV. GEOMETRIC SHRINKING: MEAN FIRST-PASSAGE TIME

In this section we will investigate the mean first-passage time of a random walker with $s_n = \lambda^n$. Since there are no generally tractable analytic methods we will resort to numerical methods in this section. Contrary to the ordinary random walk, yielding infinite mean first-passage time and hence rendering a direct Monte-Carlo analysis rather hard if not impossible [18], the walker we are interested in ultimately comes to a stop. So a numerical analysis is indeed very feasible. There are however certain issues to be resolved in the present case that do not exist in the ordinary random walk. We start this section by listing the numerical and simulation peculiarities we have encountered when studying the system. We then work out the exact soluble case of $\lambda \leq 1/2$ as a first exercise. Afterwards we present a case study for $\lambda = 0.55$ where we lay out the results for the mean first-passage time along with digressions on its various aspects. We have also present simulations for $\lambda = \sqrt{2}$ and $\lambda = 0.9$ in order to better convey the ideas and to have contrasts to the case $\lambda = 0.55$ and $\lambda \leq 1/2$.

A. Numerical peculiarities

First passage from a point. The fact that the support of the distribution evolves haunts us here again. In the ordinary random walk a first-passage from a point is actually to occupy that point for the first time. In the case of a walker with shrinking steps it is no longer possible to predict that a given point is part of the support set at a given time. However the idea of passing certainly makes sense: that is to be to the right of a point for the first time is a meaningful statement. This unfortunately has the corollary that a particular path may contribute to the first-passage distribution of more than one point if these points are close. As we will see, due to its local aspect, this effect will not alter the global properties of the system.

Falling out of range. In the ordinary random walk the

walker never falls out of range of a point. But, with the walker we are interested in this will be possible. Certain walks will fall out of range of a point due to a *bad choice of direction* at some point earlier in time. One could say earlier mistakes are punished more severely. However in terms of computer simulations this is more than welcome since it eliminates a walk that would never reach a point. This is a great help in reducing the simulation time.

Spatial resolution. A random walker with geometrically shrinking step size will theoretically walk for an infinite amount of time. However in practice one can assume that after some time it will not move appreciably and the subsequent walk will have negligible effect. We have always chosen such a cutoff time such that the subsequent step sizes would be less than 10^{-9} .

With these in mind we have performed several simulation as follows. We start the walker at the origin and make a histogram of the time values for which the walker first passes a point x_0 . This defines the first passage probability distribution $F(x_0, N)$. We then compute the normalized mean first-passage time as follows:

$$\langle t(x_0) \rangle \equiv \frac{F_1(x_0)}{F_0(x_0)}, \quad (13)$$

with

$$F_n(x_0) \equiv \sum_{N=0}^{\infty} N^n F(x_0, N), \quad (14)$$

defining the moments of $F(x_0, N)$.

The general details of the simulations were as follows. For a given x_0 , we let the walker walk 10^5 times up to the step number cutoff defined so that the spatial resolution is below 10^{-9} . After this bunch is completed we have a single distribution for $F(x_0, N)$. We repeat this procedure about 200 times to get a statistical histogram from which everything can be computed. The procedure is repeated for different values of x_0 .

B. Exact result for $\lambda \leq 1/2$

This case has a very simplifying aspect: as time passes the walker always falls out of range of any point it occupied in the past. Another way of stating this fact is that the support of the distribution is a Cantor Set which never crosses itself. Therefore the mean first-passage time will be given as follows:

$$\begin{aligned} \langle t(x_0) \rangle &= 1 && \text{for } 0 < x_0 \leq 1, \\ \langle t(x_0) \rangle &= 2 && \text{for } 1 < x_0 \leq 1 + \lambda, \\ \langle t(x_0) \rangle &= 3 && \text{for } 1 + \lambda < x_0 \leq 1 + \lambda + \lambda^2, \\ &\vdots && \end{aligned} \quad (15)$$

That is, the only walk that could pass from $0 < x_0 \leq 1$ for the first time is a walk that takes the first step to the right. Conversely if the walker chooses to go to the left in the first step

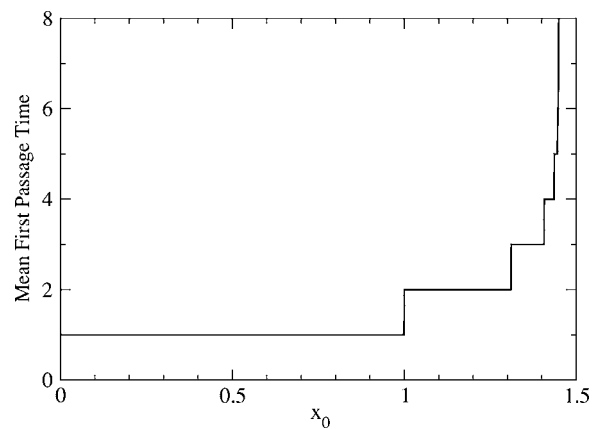


FIG. 1. The mean first-passage time from x_0 for $\lambda = \pi/10$.

it falls out of range of $0 \leq x_0 \leq 1$. The argument also carries to other intervals in (15). This result can be put to a more compact form as follows:

$$\langle t(x_0) \rangle = \text{int} \left[\frac{\ln[1 - (1 - \lambda)x_0]}{\ln \lambda} \right]. \quad (16)$$

As can be verified this function increases with unit steps as x_0 reaches certain values given by

$$x_0(K) = \frac{1 - \lambda^K}{1 - \lambda}. \quad (17)$$

These values represent what can be called *extremity walks*. That is, walks with all the steps in one direction.

Thus the mean first-passage time has the form of a ladder with step heights increasing by unity and the step widths decreasing in such a way that they are scaled down with λ at each step. In all the cases we have studied we have observed the ladder pattern, there is an appreciable discrete jump in the mean first-passage time as one crosses an extremity walk point. The reason for this pattern is that once the walker reaches a point in say k steps the subsequent walk will have the distribution $P(|x - x_0|/\lambda^k)$. However, for general λ , the step heights and widths of the ladder does not exactly follow the behavior of the ladder for $\lambda \leq 1/2$. Because as one increases λ there will be many paths from the origin to a particular x_0 . To justify the analysis for the case of $\lambda \leq 1/2$ we also present the numerical simulation for a specific case $\lambda = \pi/10$ in Fig. 1.

We also would like to make a note of the fact that for $x_0 = 1/(1 - \lambda) - \epsilon$, that is x_0 being close to its maximal range with ϵ small, we get

$$\langle t(x_0) \rangle = \frac{\ln \epsilon}{\ln \lambda}. \quad (18)$$

This has the same logarithmic divergence as (10) if we make the correspondence

$$\tau \text{ is equivalent to } -\frac{1}{\ln \lambda}. \quad (19)$$

We thus have at least a qualitative connection to the continuum case as one would expect.

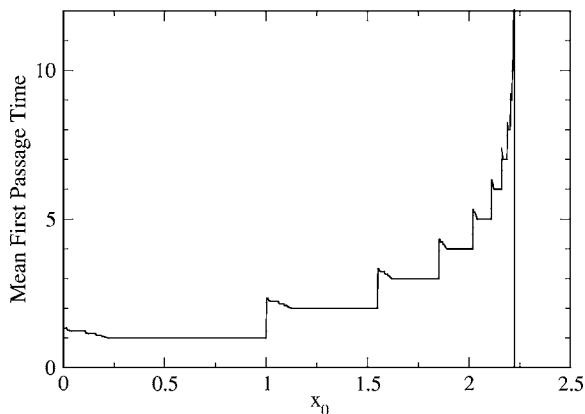


FIG. 2. The mean first-passage time from x_0 for $\lambda=0.55$. The vertical line at $x=2.\bar{2}$ is the ultimate range $1/(1-\lambda)$ of the walker for this value of λ . The error bars are very small and are omitted.

C. Case study for $\lambda=0.55$

The result for $\lambda=0.55$ can be seen in Fig. 2. As we have mentioned, the graph has a global increase pattern and also a self-repetitive, ladderlike local structure where we also perceive slight differences from the case $\lambda \leq 1/2$ (see Fig. 1) to the left of each ladder. That is the function between $x_0 \in [1, 1+\lambda]$ looks like the one between $x_0 \in [0, 1]$ but scaled with λ . This is because once we reach $x_0=1$ the subsequent walk can be seen as starting from $x_0=1$ and with an overall scale of λ appearing in front of the distances. Consequently, to have a feeling of the local behavior of the first passage time, it is sufficient to study the graph for mean first passage time for only $x_0 \in [0, 1]$.

In Fig. 3 which is a zoom out of Fig. 2 to the region $x_0 \in [0, 1]$ we see interesting features and we outline their meaning below.

(1) *Straight line between $x_0 \in [0.\bar{2}, 1]$.* This is a manifestation of the falling out of range effect we have outlined above. The error bar of this line vanishes and this can be due to only one reason: there is one and only one path that can pass from this interval [19]. This becomes easier to understand if we also realize that the value $0.\bar{2}$ is nothing but

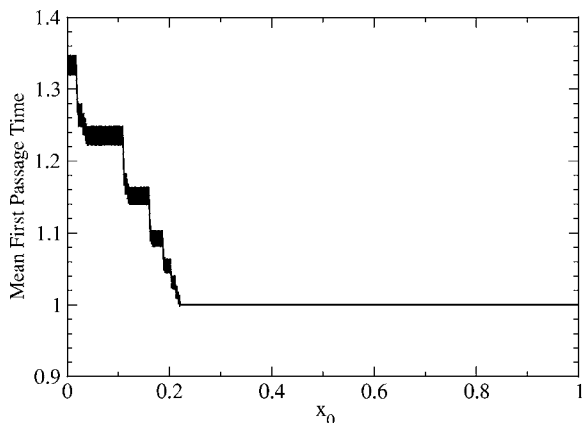


FIG. 3. Zoom of Fig. 2 to the region $x_0 \in [0, 1]$. The straight line has zero error. See text for explanation.

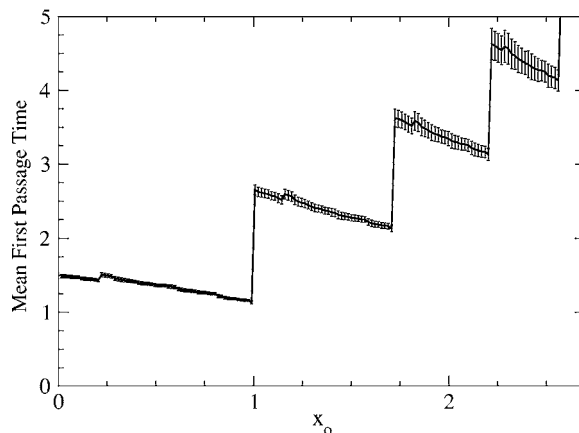


FIG. 4. Mean first-passage time for $\lambda=1/\sqrt{2}$ for the first few steps in the ladder.

$-1 + \lambda/(1-\lambda)$ for $\lambda=0.55$. Thus if the walker chooses to take the first step to the left then the ultimate position it can go is simply the value quoted above. Any point beyond this value can only be first passed by a single step to the right. One could now ask: what should be the largest value for λ such that there is such a behavior? This is answered by the solution of the inequality

$$-1 + \frac{\lambda}{1-\lambda} \leq 1 \rightarrow \lambda \leq \frac{2}{3}. \tag{20}$$

It is instructive to note that this value of λ has been observed in [8] to be the smallest value for which there are an infinite number of paths from the origin to any point. This, we would like to contrast to the fact that the requirement to have at least one path from the origin to any point, is satisfied for $\lambda \geq 1/2$, again mentioned in [8]. Here we see an example where the condition is reversed. That is, there are only finite number of paths to certain positions and some positions can only be reached by unique paths. For $\lambda \geq 2/3$ we expect this straight line behavior to disappear since there will be an infinite number of paths from the origin to any point. See Fig. 4 and then see Fig. 5 for $\lambda=1/\sqrt{3}$ and Fig. 6 for $\lambda=0.9$ both of which have $\lambda \geq 2/3$.

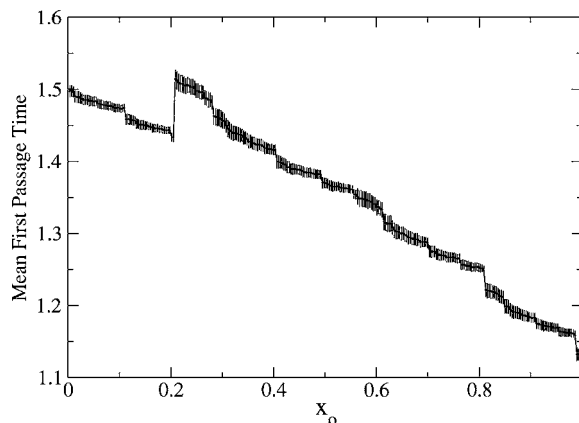


FIG. 5. Mean first-passage time for $\lambda=1/\sqrt{2}$ for the first step of the ladder.

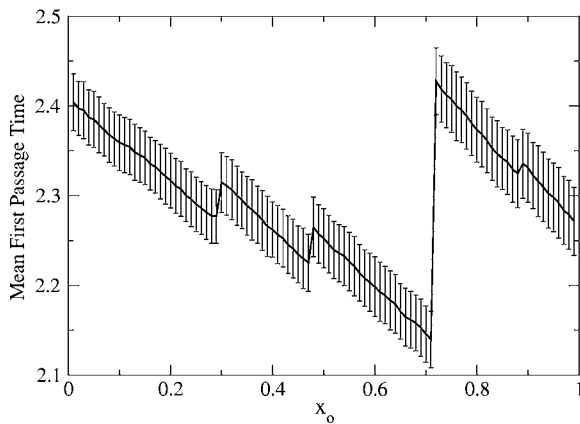


FIG. 6. Mean first-passage time for $\lambda=0.9$ for the first step of the ladder.

(2) *There are multiple plateaus.* This is again a manifestation of the condition that for $\lambda \geq 2/3$ there are infinite paths from the origin to any point. The converse is not necessarily true and it is apparent in Fig. 3. Certain values of x_0 are reached by a *finite* (not one) number of paths and there are ranges of x_0 such that this number is constant [20]. Consequently, the appearance of plateaus is guaranteed by the condition $\lambda \leq 2/3$ and we see this to disappear in Figs. 5 and 6 (both of which represent cases for $\lambda \geq 2/3$) where the plateau structure has transformed into a cascaded hill form.

(3) *Time goes up as x_0 gets small?* If we consider Figs. 2 and 3, we observe that as opposed to the global increase of the ladder, the mean first-passage time has a local increase pattern if we go to low x_0 . This local increase, as we have outlined above, is due to the fact that one particular path might contribute to the first-passage distribution of more than one x_0 , and as $x_0 \rightarrow 0$ the number of paths that do so generally increase without decreasing the probability of the first-passage distribution. On the other hand we expect as $\lambda \rightarrow 1$ that this behavior will disappear since the number of x_0 's, a particular path might contribute, would naturally decrease.

D. Intermediate conclusions

Here we would like to sum up what we have observed so far. For each λ the curve for the mean first-passage time has the form of a ladder. The ladder has appreciable jumps at the extremity walk points $x_0(K)$ given by (17). These jumps are forming the backbone of the global increase as x_0 gets larger. However on a particular step of the ladder, that is for $x_0(K) \leq x_0 \leq x_0(K+1)$, there is very rich local behavior. We have studied some of these. On the other hand, this local behavior is not very fierce. For example, if we consider the first step of the ladder for different λ 's as in Figs. 3, 5, and 6, we see that the mean first-passage time is not too different than the value to the rightmost value of the step. Therefore to study the global increase pattern it should be enough to consider only the first-passage time from extremity walk points $x_0(K)$.

V. GLOBAL STRUCTURE FOR GENERAL λ

For $\lambda \geq 2/3$ the mean first-passage time differs from (16) for the extremity walks defined in (17) as can be checked

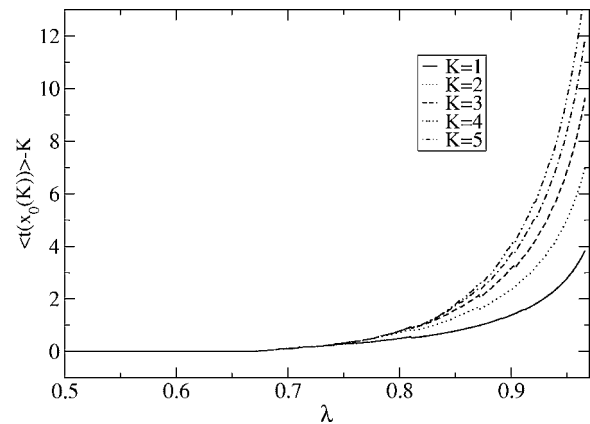


FIG. 7. λ behavior of the first-passage time for various extremity walk points. Error bars are omitted for clarity.

from Figs. 5 and 6 where the mean first-passage times are not 1 for $x_0(1)=1$. Nevertheless, as we have mentioned, with increasing x_0 the ladder structure is still present, and the ladder jumps are still occurring at $x_0(K)$. So we have to consider another approach. To study how much discrepancy arises, we have simulated the random walk for different extremity points and computed the mean first-passage time minus the values that is given by (16), that is for each λ we computed

$$\langle t'(K) \rangle \equiv \langle t(x_0(K)) \rangle - K. \tag{21}$$

The result is presented in Fig. 7, where we have again suppressed the error bars to have a clear picture. A representative picture with the error bars is given in Fig. 8.

The discrepancy starts at $\lambda=2/3$ as anticipated. It is small for not too large values of λ and diverges later on. However we see that the discrepancy itself has a pattern. All the curves agree up to a certain value of λ , then the curve for $K=1$ starts to deviate. After this point the remaining ones agree up to another certain value of λ after which the curve for $K=2$ starts to deviate. This goes on in the same way for higher values of K .

To understand and quantify this behavior we would like to remind the reader that extremity walks are in essence the

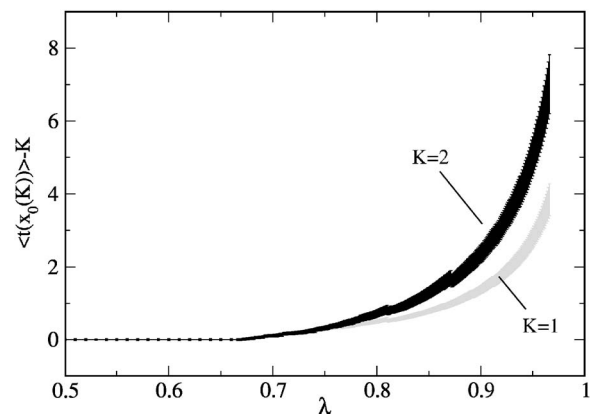


FIG. 8. Figure 7 shown with error bars for the first two extremity walk points.

fastest paths to a given point. Then, the next to fastest path would be one with a single step in the wrong direction. Of course it is important when the wrong step is realized and this will be the crucial point in our line of reasoning. Let us try to construct our argument by an example: consider a first-passage path from the origin to $x_0(2)$, the fastest path is of course an extremity walk that would take two steps to the right. Next, we would like to consider paths with one step in the wrong direction that would still pass from $x_0(2)$. There are two possibilities: one can either choose the wrong direction in the first step or in the second. Considering these choices and demanding the subsequent walk would be in range, we have two equations

$$-1 + \frac{\lambda}{1-\lambda} = 1 + \lambda \rightarrow \lambda \approx 0.732\,051, \quad (22)$$

$$1 - \lambda + \frac{\lambda^2}{1-\lambda} = 1 + \lambda \rightarrow \lambda = \frac{2}{3}. \quad (23)$$

Now (23) is the same as the condition to reach $x_0(1)$ with one step in the wrong direction, so it is not a new condition. However (22) is new and once λ gets bigger than the value quoted, this walk [which was already contributing to $x_0(1)$] will start to contribute to the first-passage time for $x_0(2)$. Thus between these two values of λ , $\langle t'(K) \rangle$ will be the same for $K=1$ and $K=2$, and consequently for all K . However as soon as λ gets bigger than 0.732 051 a new path will start to contribute to the first-passage time for $x_0(2)$, meaning that its difference from the base value 2 will be bigger. This is why the curves for $K=1$ and the rest start to branch at this point.

To iterate and make the idea clearer let us now consider the same argument for $x_0(3)$. There are three possible ways one can opt for the wrong direction and demanding the walks will be in range we get the following:

$$-1 + \frac{\lambda}{1-\lambda} = 1 + \lambda + \lambda^2, \quad (24)$$

$$1 - \lambda + \frac{\lambda^2}{1-\lambda} = 1 + \lambda + \lambda^2, \quad (25)$$

$$1 + \lambda - \lambda^2 + \frac{\lambda^3}{1-\lambda} = 1 + \lambda + \lambda^2, \quad (26)$$

which would yield, respectively

$$\lambda \approx 0.770\,917, \quad (27)$$

$$\lambda \approx 0.732\,051, \quad (28)$$

$$\lambda = \frac{2}{3}. \quad (29)$$

We have indeed numerically observed the fact that the curves for $K=2$ and $K=3$ agree up to around $\lambda \approx 0.770\,917$. Since as λ gets bigger than this value a path that was already contributing to $x_0(1)$ and $x_0(2)$ would be added to the path list of $x_0(3)$ and hence creating a shift in the curve.

It is also possible to argue about the existence of paths with two or more steps in the wrong direction which will in

TABLE I. Possible roots of Eq. (30) for the first few values of (K, P) are presented.

$P \setminus K$	1	2	3
1	0.666667	0.732051	0.770917
2	0.780776	0.816497	0.839287
3	0.835122	0.858094	0.87358

general give an infinite number of possible constraint equations and possibly interfering with the picture above. Let us consider the case with two steps in the wrong direction. There are three generally possible cases. First, one could choose the wrong steps early in the walk and not so separated in time thus subjecting the subsequent walk to greater punishment. Second, one could choose them after having subsequent steps to the right, again not so separated in time, meaning a much less punishment for the subsequent walk. Or third, one could have them separated by a long walk to the right for which the punishment would be somewhere in between compared to the other two cases. It is clear that only the first case comes with more probability because less number of steps are fixed [21]. So we would like to introduce the idea of *worst case punishment extremity walk* defined as the path in which all the wrong steps are taken at the beginning of the walk. This is a generalization of the idea we outlined above. The general constraint equation for the P -step worst case punishment extremity walk (that is P steps to the left at the very beginning) to go to $x_0(K)$ extremity point is given by

$$\lambda^K + 2\lambda^P - 2 = 0. \quad (30)$$

The possible roots of this equation for the first few values of (K, P) are presented in Table I.

From Table I the fourth highest value is given for $(K=1, P=2)$ and not $(3, 1)$. This means that just before the path $(3, 1)$ contributes to $x_0(3)$ the path $(1, 2)$ starts to contribute to $x_0(1)$ and this condition would also satisfy a different walk of the form $1 - \lambda - \lambda^2 + \lambda^3 / (1 - \lambda) = x_0(2) = 1 + \lambda$. So the process of branching we have mentioned above is actually more complex. However the behavior from $\lambda = 0.666\,66$ to $\lambda = 0.732\,051$ is free of these complexities. Thus one can safely state that we have the following:

$$\langle t'(K) \rangle = f(\lambda) \quad \text{for } \frac{2}{3} \leq \lambda \leq 0.732\,051, \quad (31)$$

a function independent of K .

As a final remark we would like to mention that all the curves in Fig. 7 fit very well to the following ansatz:

$$\frac{a_0(K)}{\sqrt{-\ln \lambda}} + a_1(K) + \frac{a_2(K)}{\ln \lambda}, \quad (32)$$

with a_0 , a_1 , and a_2 are fit parameters. Here the first two terms, which dominates the small λ behavior, represent an analogy the small u behavior of (9), whereas the last one, which dominate the large λ behavior, is the analog of large u asymptotic behavior in (10), if we would like to make the qualitative identification $\tau \sim -1/\ln \lambda$. Since, with this identi-

fication and for fixed $x_0(K)$, increasing (decreasing) λ would be similar to increasing (decreasing) u . So the discrete case is not qualitatively very different than the continuum model we have presented.

VI. CONCLUSION

In this work we have studied the first-passage characteristics of a random walker with step sizes decaying exponentially in discrete time. We have also shown that the discrete case shares all the qualitative properties of a continuum diffusion equation with an exponentially decaying diffusion constant which hints phenomenologically that the continuum case might be of choice for studying general aspects of such physical systems. While the discrete case shares global qualitative properties of the continuum model it locally shows very rich mathematical structure, the most important of these is that the mean first-passage time undergo transitions at λ 's defined by the equation $\lambda^K + 2\lambda^P - 2 = 0$ with $\{K, N\} \in \mathcal{N}$. In order not to repeat ourselves we refer the reader to the end of the Introduction for a short review of these local properties.

Although, mainly to connect to the literature, we have confined ourselves to the exponentially decaying step sizes, there are many other possibilities. One possible interesting example that comes to mind is to assume $s_N = (1 - N/M)^\alpha$ with M integer and α arbitrary. This case has the nice extra feature that the walk really comes to a stop when the walker commits M steps therefore allowing an exact enumeration of paths for small values of M . Furthermore this example has a connection to the exponential case remembering $e^x = \lim_{a \rightarrow \infty} (1 + x/a)^a$. A preliminary analysis we have carried out gives similar behavior to the exponential case while differing in details. Another likely extension is, as proposed in [7], the n -dimensional random walker with shrinking steps which might provide further interesting features.

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 - [17] This argument is presented also in [2].
 - [18] We mean a direct simulation would be hard in the ordinary random walk for large times. One can, in principle, study the first-passage time by other numerical methods such as measuring the flux to a point. Such an alternate approach would be more feasible than a direct analysis.
 - [19] One might argue that in a computer simulation a zero error bar can be a fake one if one does not cover the probabilities accurately. This is not the case here and as the text explains the proposition is rigorous.
 - [20] Since the number of paths is not "one" we have error bars for these cases.
 - [21] This line of reasoning about the probability of paths might seem speculative at first since we actually let the walker take an infinite step to the right after all the wrong steps are taken. However this is only a constraint equation, once λ gets bigger than the value given by the constraint, the last phase of the walk will certainly consist of a finite number of steps with step number decreasing with increasing λ . But the number of steps fixed at the beginning would still be reducing the overall probability of the path in question.